論　文

The number of k-simplices in the barycentric subdivision of an n-simplex

n-単体を重心細分したときにあらわれるk-単体の個数

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概要：「単体」とは，点（0-単体），線分（1-単体），三角形（2-単体），四面体（3-単体）を一般化したものである。空間の基本単位となっている。「重心細分」とは，包含関係にある面同士の重心を結ぶことにより，凸多面体を同じ次元の単体に分割する手法である。この2つの用語は，位相幾何学において，重要な意味を持つ。本論文の目的は，n-単体を重心細分したとき，あらわれるk-単体の個数を求めることである。重心細分をした単体複体を，元のn-単体の内部と境界に分けることにより，帰納法を適用して求めた。

1. Introduction

Topology is a new field in mathematics whose concept was established in the 20th century. Homology which defined by H. Poincarè [1] is one of the important tools of topology. What is needed to define homology is a simplex. There are several theorems that use the simplices, and one of the important theorems is the simplicial approximation theorem [2]. Any continuous maps between polyhedra are approximated as a morphism of simplices via the barycentric subdivision of the polyhedra.

Although simplices and its barycentric subdivisions are basic terms, almost no mention is made of the number of simplices. The purpose of this research is to determine calculate the number of k-simplices in the barycentric subdivision of an n-simplex.

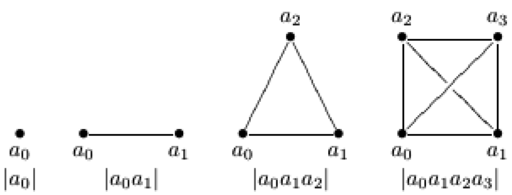
2. Main Theorem

Definition 2.1. (simplices) Let N and n be natural numbers with $n \leq N$, and $a_0,\ a_1,..., a_n \in \mathbb{R}^N$. If the set of vectors $a_1 - a_0,...,\ a_n - a_0$ is linearly independent, then we call the following an *n-simplex.*

$$\sigma = |a_0 a_1 \cdots a_n|$$

$$= \left\{ \sum_{i=0}^{n} \lambda_i a_i \in \mathbb{R}^N \mid \sum_{i=0}^{n} \lambda_i = 1, \lambda_i \geq 0 \right\}$$

For example, a 0-simplex, a 1-simplex, a 2-simplex, a 3-simplex are a point, a line segment, a triangle and a tetrahedra, respectively.



The standard *n-simplices* Δ^n is defined by,

$$\Delta^n = \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1 \text{ and } x_i \geq 0 \right\}.$$

Let $\sigma = |a_0 a_1 \cdots a_n|$ be an n-simplex. Then for all subset $\{a_{i_0}, a_{i_1}, \dots, a_{i_m}\}$ of $\{a_0, a_1, \dots, a_n\}$,

$$\tau = |a_{i_0} a_{i_1} \cdots a_{i_m}|,$$

is also simplex. In this case, τ is called *faces* of σ and writen by $\tau \prec \sigma$.

Definition 2.2. (simplicial complexes) A set K of simplices is a *simplicial complex* if and only if K saticefies the followings:

(1) $\sigma \in K, \tau \prec \sigma \implies \tau \in K.$

(2) $\sigma, \tau \in K, \sigma \cap \tau \neq \phi \implies \sigma \cap \tau \prec \sigma, \sigma \cap \tau \prec \tau.$

A *polyhedron* of simplicial complexes, denotes as $|K|$, is the union of its simplices.

$$|K| = \bigcup_{\sigma \in K} \sigma .$$

Definition 2.3. (barycentric subdivisions) Let $\sigma = |a_0 a_1 \cdots a_n|$ be an n-simplex. The *centroid* b_σ of σ is denoted by

$$b_\sigma = \frac{1}{n+1} \sum_{i=0}^{n} a_i .$$

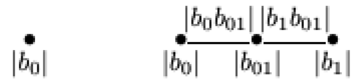
The *barycentric subdivision* of σ is the following simplicial complex:

$$\mathrm{Sd}(\sigma) = \left\{ |b_{\sigma_0} \cdots b_{\sigma_m}| \,\middle|\, \phi \neq \sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_m \prec \sigma \right\}$$

Example 2.4.

(1) $\mathrm{Sd}(|a_o|) = \{|b_0|\}$,

(2) $\mathrm{Sd}(|a_o a_1|) = \{|b_0|, |b_1|, |b_{01}|, |b_0 b_{01}|, |b_1 b_{01}|\}$,



where $b_0 = a_0$, $b_1 = a_1$ and $b_{01} = \frac{1}{2}(a_0 + a_1)$.

We define the number $\#\mathrm{Sd}\Delta^n(k)$ to be the number of k-simplices in the barycentric subdivision of Δ^n. The purpose of this paper is to determine $\#\mathrm{Sd}\Delta^n(k)$.

Theorem 2.5. *The number of the k-simplices in the barycentric subdivision of the n-simplex Δ^n is*

$$\sum_{i=1}^{k+2} (-1)^{k+i} \binom{k+1}{i-1} i^{n+1} .$$

Example 2.6. We understand the followings from figures.

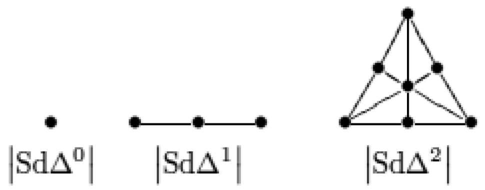


Table 1: $\#\mathrm{Sd}\,\Delta^n(k)$

$n\backslash k$	0	1	2	3
0	1			
1	3	2		
2	7	12	6	
3	15	50	60	24

Via the binomial theorem, it is easy to show the following example.

Example 2.7. For any n,

$$\#\mathrm{Sd}\Delta^n(0) = 2^{n+1} - 1 .$$

3. Proof of Main Theorem

The We define the number $\#\mathrm{Sd}\,\partial\Delta^n(k)$ and $\#\mathrm{Sd\,Int}\Delta^n(k)$ to be the number of k-simplices of $\mathrm{Sd}\Delta^n$ which appers in the boundary of Δ^n and inside of Δ^n, respectively. This notation means

$$\#\mathrm{Sd}\Delta^n(k) = \#\mathrm{Sd}\partial\Delta^n(k) + \#\mathrm{Sd\,Int}\Delta^n(k)$$

Table 2: $\#\mathrm{Sd}\,\partial\,\Delta^n(k)$

$n\backslash k$	0	1	2	3
0				
1	2			
2	6	6		
3	14	36	24	

Table 3: $\#\mathrm{Sd\,Int}\,\Delta^n(k)$

$n\backslash k$	0	1	2	3
0	1			
1	1	2		
2	1	6	6	
3	1	14	36	24

Moereover, formally defined as

$$\#\mathrm{Sd}\,\partial\Delta^n(-1) = 1.$$

In the interior of Δ^n, there exists a unique 0-simplex of $\mathrm{Sd}\Delta^n$. Therefore the number of k-simplices of $\mathrm{Sd}\Delta^n$ inside Δ^n matches the number of $(k-1)$-simplices of $\mathrm{Sd}\Delta^n$ in the boundary of Δ^n. In other words, for any $0 \leq k \leq n$

$$\#\mathrm{Sd\,Int}\Delta^n(k) = \#\mathrm{Sd}\partial\Delta^n(k-1) .$$

On the other hand, consider the number of the k-simplices of $\mathrm{Sd}\Delta^n$ in the boundary of Δ^n.

$$\#\mathrm{Sd}\partial\Delta^n(k) = \sum_{j=k}^{n-1} \binom{n+1}{j+1} \#\mathrm{Sd\,Int}\Delta^j(k)$$

$$= \sum_{j=k}^{n-1} \binom{n+1}{j+1} \#\mathrm{Sd}\partial\Delta^j(k-1),$$

where $\binom{n+1}{j+1}$ denotes a binomial coefficient.

In order to prove Theorem 2.5., we use the following proposition.

Proposition 3.1. *For any* $1 \leq k \leq n$, *define*

$$A(n,k) = \begin{cases} 1, & (k=1) \\ \sum_{j=k-1}^{n-1} \binom{n}{j} A(j,k-1), & (k \geq 2) \end{cases}.$$

Then we obtain

$$A(n,k) = \sum_{i=1}^{k} (-1)^{k+i} \binom{k}{i} i^n .$$

In this case, $A(n,k)$ defines as follows.

$$A(n,k) = \#\mathrm{Sd}\partial\Delta^{n-1}(k-2)$$

Actually

$A(n,k)$

$$= \#\mathrm{Sd}\partial\Delta^{n-1}(k-2)$$

$$= \begin{cases} 1, & (k=1) \\ \sum_{j=k-2}^{n-2} \binom{n}{j+1} \#\mathrm{Sd}\partial\Delta^{j}(k-3), & (k \geq 2) \end{cases}$$

$$= \begin{cases} 1, & (k=1) \\ \sum_{j=k-1}^{n-1} \binom{n}{j} \#\mathrm{Sd}\partial\Delta^{j-1}(k-3), & (k \geq 2) \end{cases}$$

$$= \begin{cases} 1, & (k=1) \\ \sum_{j=k-1}^{n-1} \binom{n}{j} A(j,k-1), & (k \geq 2) \end{cases}.$$

Then we can apply Proposition 3.1. to this case.

$\#\mathrm{Sd}\Delta^n(k)$

$$= \#\mathrm{Sd}\partial\Delta^n(k) + \#\mathrm{Sd}\partial\Delta^n(k-1)$$

$$= A(n+1,k+2) + A(n+1,k+1)$$

$$= \sum_{i=1}^{k+2} (-1)^{k+i+2} \binom{k+2}{i} i^{n+1} + \sum_{i=1}^{k+1} (-1)^{k+i+1} \binom{k+1}{i} i^{n+1}$$

$$= \sum_{i=1}^{k+2} (-1)^{k+i} \left\{ \binom{k+2}{i} - \binom{k+1}{i} \right\} i^{n+1}$$

$$= \sum_{i=1}^{k+2} (-1)^{k+i} \binom{k+1}{i-1} i^{n+1},$$

where $\binom{k+1}{k+2} = 0$. We have the result.

4. Proof of Proposition 3.1.

In this section, we prove Proposition 3.1. First, we prepare the following lemma.

Lemma 4.1. *For any* $0 \leq j \leq k-1$,

$$\sum_{i=1}^{k} (-1)^i \binom{k}{i} i^j = \begin{cases} 1, & (j=0) \\ 0, & (j \leq 1) \end{cases}.$$

Proof. When $j=0$,

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} = 0.$$

induces the claim.

Next, we consider the case of $j \geq 1$. Assume that

$$\sum_{i=1}^{k} (-1)^i \binom{k}{i} i^{\ell} = 0, \quad (0 \leq \ell \leq k-1).$$

Then for any $1 \leq j \leq k$,

$$\sum_{i=1}^{k+1} (-1)^i \binom{k+1}{i} i^j$$

$$= \sum_{i=1}^{k+1} (-1)^i \binom{k}{i-1} i^j + \sum_{i=1}^{k} (-1)^i \binom{k}{i} i^j \quad \cdots (*)$$

$$= \sum_{i=0}^{k} (-1)^{i+1} \binom{k}{i} (i+1)^j + \sum_{i=1}^{k} (-1)^i \binom{k}{i} i^j$$

$$= -1 - \sum_{i=1}^{k} (-1)^i \binom{k}{i} \left\{ (i+1)^j - i^j \right\}$$

$$= -1 - \sum_{i=1}^{k} \sum_{s=0}^{j-1} (-1)^{i+j-s-1} \binom{k}{i} (i+1)^s i^{j-s-1}$$

$$= -1 - \sum_{i=1}^{k} \sum_{s=0}^{j-1} \sum_{t=0}^{s} (-1)^{i+j-s-1} \binom{k}{i} \binom{s}{t} i^{j-s+t-1}$$

$$= -1 - \sum_{s=0}^{j-1}\sum_{t=0}^{s}(-1)^{j-s-1}\binom{s}{t}\sum_{i=1}^{k}(-1)^{i}\binom{k}{i} i^{j-s+t-1}.$$

Because $0 \le j-s+t-1 \le j-1 \le k-1$, we see that $j-s+t-1=0$ if and only if $(s,t)=(j-1,0)$. Then

$$\sum_{i=1}^{k}(-1)^{i}\binom{k}{i} i^{j-s+t-1} = \begin{cases} -1, ((s,t)=(j-1,0)) \\ 0, ((s,t)\neq(j-1,0)) \end{cases}.$$

Therefore,

$$\sum_{i=1}^{k+1}(-1)^{i}\binom{k+1}{i} i^{j}$$

$$= -1 - \sum_{s=0}^{j-1}\sum_{t=0}^{s}(-1)^{j-s-1}\binom{s}{t}\sum_{i=1}^{k}(-1)^{i}\binom{k}{i} i^{j-s+t-1}$$

$$= -1-(-1)$$

$$= 0.$$

(q.e.d.)

Remark 4.2. See $(*)$ in the proof of Lemma 4.1. Note that when $j=k$,

$$\sum_{i=1}^{k}(-1)^{i}\binom{k}{i} i^{j} \neq 0.$$

Proof of Proposition 3.1. We prove the assertion as introduction argument on n. Suppose $2 \le k \le n$ and the statement holds for less than $n=1$. Then we obtain

$$A(n,k)$$

$$= \sum_{j=k-1}^{n-1}\binom{n}{j} A(j,k-1)$$

$$= \sum_{j=k-1}^{n-1}\sum_{i=1}^{k-1}(-1)^{k+i-1}\binom{n}{j}\binom{k-1}{i} i^{j}$$

$$= \sum_{i=1}^{k-1}(-1)^{k+i-1}\binom{k-1}{i}\left(\sum_{j=k-1}^{n-1}\binom{n}{j} i^{j}\right)$$

$$= \sum_{i=1}^{k-1}(-1)^{k+i-1}\binom{k-1}{i}\left((i+1)^{n} - i^{n} - \sum_{j=0}^{k-2}\binom{n}{j} i^{j}\right)$$

$$= \sum_{i=1}^{k-1}(-1)^{k+i-1}\binom{k-1}{i}(i+1)^{n}$$

$$- \sum_{i=1}^{k-1}(-1)^{k+i-1}\binom{k-1}{i} i^{n}$$

$$- \sum_{i=1}^{k-1}\sum_{j=0}^{k-2}(-1)^{k+i-1}\binom{k-1}{i}\binom{n}{j} i^{j}$$

$$= \sum_{i=2}^{k}(-1)^{k+i}\binom{k-1}{i-1} i^{n} + \sum_{i=1}^{k-1}(-1)^{k+i}\binom{k-1}{i} i^{n}$$

$$+ \sum_{j=0}^{k-2}(-1)^{k}\binom{n}{j}\sum_{i=1}^{k-1}(-1)^{i}\binom{k-1}{i} i^{j}$$

$$= \sum_{i=2}^{k}(-1)^{k+i}\binom{k}{i} i^{n} + (-1)^{k+1}\binom{k-1}{1} + (-1)^{k+1}$$

$$= \sum_{i=2}^{k}(-1)^{k+i}\binom{k}{i} i^{n} + (-1)^{k+1}\binom{k}{1}$$

$$= \sum_{i=1}^{k}(-1)^{k+i}\binom{k}{i} i^{n}.$$

(q.e.d.)

A. Appendix

By definition $A(n,k)$ written in Proposition 3.1., we have

$$A(n,n) = \binom{n}{n-1} A(n-1,n-1).$$

Then,

$$\sum_{i=1}^{n}(-1)^{n+i}\binom{n}{i} i^{n} = n!.$$

Therefore, Lemma 4.1. is expanded to the following.

Proposition A.1. *For any* $k \ge 1, 0 \le j \le k$,

$$\sum_{i=1}^{k}(-1)^{i}\binom{k}{i} i^{j} = \begin{cases} 1, & (j=0) \\ 0, & (1 \le j \le k-1) \\ (-1)^{k} k!, & (j=k) \end{cases}.$$

Acknowledgement

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References

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